Approximating the Hurwitz Zeta Function

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Presentation

The Riemann Zeta Function

Definition

The **Riemann Zeta function** $\zeta(s)$ is defined for complex inputs s as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Hurwitz Zeta Function

The Hurwitz Zeta function generalizes the Riemann Zeta.

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Definition

The **Hurwitz Zeta function** $\zeta(s, a)$ is defined for complex inputs s and a with $Re(a) > 0$ and $Re(s) > 1$ as follows:

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

The Hurwitz Zeta Function

The Hurwitz Zeta function generalizes the Riemann Zeta.

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Definition

Compare this to the definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Hurwitz Zeta Function

The Hurwitz zeta function can be analytically continued to almost all complex arguments $\mathrm{Re}(s) \leq 1$ as follows:

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Theorem

$$\zeta(s, q) = \Gamma(1 - s)\frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{qz}}{1 - e^z}dz.$$

The Hurwitz Zeta Function

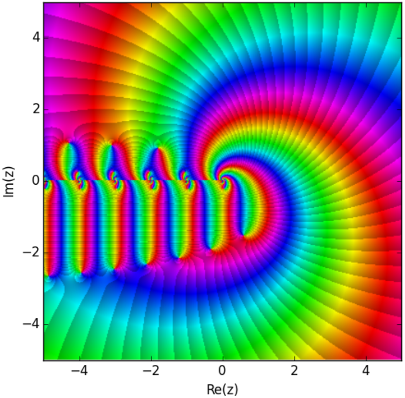

Figure: A graph of the Hurwitz zeta function as a function of a with $s = 3 + 4i$.

The Hurwitz Zeta Function

Recall the definition of the Hurwitz Zeta funtion:

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Query

How can we approximate the Hurwitz Zeta function for any satisfactory inputs s and a to arbitrary precision?

Background

Definition

The **Lerch Transcendent** $\Phi(s, a, z)$ is defined for complex inputs s,a,z as

$$\Phi(s, a, z) = \sum_{n=1}^{\infty} \frac{z^n}{(n + a)^s}.$$

Background

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$$\Phi(s, a, z) = \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^s}.$$

Definition

Once again, recall that the Hurwitz Zeta function $\zeta(s, a)$ is defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Background

Definition

1. The **gamma function** $\Gamma(s)$ is defined by the integral

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

2. The **upper incomplete gamma function** $\Gamma(s, z)$ is defined by the integral

$$\Gamma(s, z) = \int_z^\infty t^{s-1} e^{-t} dt$$

The incomplete gamma function generalizes the gamma function.

Alternate Series for the Hurwitz Zeta Function

Theorem (Bailey–Borwein, 2015)

Let λ be a parameter with $0 < \lambda < 2\pi$. Define $\sigma(x)$ to be the sign function. Then for real a and complex s with $0 < a < 1$ and $Re(s) > 1$, we have

$$\zeta(s,a) = \frac{\sqrt{\pi}\lambda^{\frac{s-1}{2}}}{(s-1)\Gamma\left(\frac{s}{2}\right)} + \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{|n+a|^s}\left(\frac{\Gamma(\frac{s}{2}, \lambda(n+a)^2)}{\Gamma(\frac{s}{2})} + \sigma(n+a)\frac{\Gamma(\frac{s+1}{2}, \lambda(n+a)^2)}{\Gamma(\frac{s+1}{2})}\right) + \pi^{s-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{1}{m^{1-s}}\left(\frac{\Gamma(\frac{1-s}{2}, \frac{m^2\pi^2}{\lambda})}{\Gamma(\frac{s}{2})}\cos(2\pi ma) + \frac{\Gamma(1-\frac{s}{2}, \frac{m^2\pi^2}{\lambda})}{\Gamma(\frac{s+1}{2})}\sin(2\pi ma)\right).$$

Our Research

Difficulties in Approximating the Hurwitz Zeta Function

Large Imaginary Parts

Set $a = e^{m+pi}$, where $0 \leq p < 2\pi$, for brevity. Consider just the first term in our summation, $\frac{1}{a^s}$.

$$\left|\frac{1}{a^s}\right| = \frac{e^{p \cdot \mathrm{Im}(s)}}{|a|^{\mathrm{Re}(s)}}.$$

Thus, when $|a|^{\mathrm{Re}(s)} \ll 1$ or $p \cdot \mathrm{Im}(s) \gg 1$, $\frac{1}{a^s}$ may grow very large in magnitude.

Analyzing Convergence

Theorem

When $a > 0$ and N some (presumably large) positive integer,

$$\left| \sum_{n=N+1}^{\infty} (n+a)^{-s} \right| < \frac{N^{1-Re(s)}}{Re(s)-1}.$$

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Idea

For real s, the series $\sum_{n=N}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

Corollary

To achieve k digits of precision in $\zeta(s,a)$, we need $O(10^{\frac{k}{Re(s)-1}})$ terms as $k \to \infty$.

Analyzing Convergence

Theorem

For real s and a with $s > 1$ *and* $a > 0$, *and integer n so that* $|n+a| \geq \frac{s}{2}$ *and* $|n+a| \geq 10$,

$$\Gamma\left(\frac{s}{2}, \pi(n+a)^2\right) < 10^{-(n+a)^2}.$$

Analyzing Convergence

Theorem

For real s and a with $s > 1$ and $a > 0$, and integer n so that $|n + a| \geq \frac{s}{2}$ and $|n + a| \geq 10$,

$$\Gamma\left(\frac{s}{2}, \pi(n+a)^2\right) < 10^{-(n+a)^2}.$$

Corollary

For any given ordered pair (s, a), we need $O(\sqrt{k})$ terms to obtain k digits of precision in $\zeta(s, a)$

Future Research

- Analyze the performance of other series
- Expand the scope of our analyses to complex s and/or a.
- Optimize Implementation

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